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Research Article

Some characterizations of BMO spaces via commutators of maximal functions on Morrey-Lorentz spaces

Heng Yang, Jiang Zhou*

College of Mathematics and System Science, Xinjiang University, Urumqi, Xinjiang 830017, China

Abstract

In this paper, we investigate the commutators of the fractional maximal function and the sharp maximal function on Morrey-Lorentz spaces. Furthermore, we present some new characterizations of BMO spaces.

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1. Introduction and main results

Let T be the classical singular integral operator, the commutator $[b,T]$ generated by T and a suitable function b is given as

$$[b,T]f(x) = bTf(x) - T(bf)(x).$$

A significant conclusion of Coifman, Rochberg and Weiss[5] showed that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson[16] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via the commutator $[b,T]$ and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ $(0 < \beta < 1)$ if and only if $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$. Recently, many authors have conducted extensive studies on the theory of commutators, as it plays a important role in harmonic analysis and partial differential equations, see for example [6, 8, 19, 20, 23].

As usual, let $B := B(x,r)$ denote the ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. We define $|B|$ as the Lebesgue measure of the ball B and let χ_B represent the characteristic function of the ball B. Define $L^1_{\text{loc}}(\mathbb{R}^n)$ as the set of all locally integrable functions on $\mathbb{R}^n$. For $1 \le p < \infty$, we define the conjugate index of p as $p' = \frac{p}{p-1}$. We will use the symbol C to refer to a positive constant that is independent of the main parameters, but it may vary from line to line. The notation $f \lesssim g$ indicates that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$ we write $f \approx g$.

*Corresponding Author.
Email addresses: yanghengxju@yeah.net (H. Yang), zhoujiang@xju.edu.cn (J. Zhou)
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Let $0 \le \alpha < n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the fractional maximal function $M_\alpha(f)$ is defined as follows:

$$M_\alpha(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.

When $\alpha = 0$, $M_0(f)$ corresponds to the classical Hardy-Littlewood maximal function. For $0 < \alpha < n$, $M_\alpha(f)$ represents the classical fractional maximal function.

The sharp maximal function $M^\sharp(f)$ was introduced by Fefferman and Stein [9] and is defined as follows:

$$M^\sharp(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x and $f_B := \frac{1}{|B|}\int_B f(x)dx$.

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the maximal commutator of the fractional maximal function $M_\alpha(f)$ is defined by

$$M_{\alpha,b}(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(x) - b(y)||f(y)|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.

The nonlinear commutator of fractional maximal function $M_\alpha(f)$ is given as

$$[b, M_\alpha](f)(x) = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x).$$

For $\alpha = 0$, we simply write by $M_b = M_{0,b}$ and $[b, M] = [b, M_0]$.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \ge 0. \\ |b(x)|, & \text{if } b(x) < 0. \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Clearly, $b(x) = b^+(x) - b^-(x)$.

Let $b \ge 0$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\begin{aligned} |[b, M_\alpha] f(x)| &= |b(x)M_\alpha f(x) - M_\alpha(bf)(x)| \\ &= \Big| b(x) \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|dy - \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(y)f(y)|dy \Big| \\ &\le \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(x) - b(y)||f(y)|dy \\ &= M_{b,\alpha}(f)(x). \end{aligned}$$

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then, for $x \in \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$|[b, M_\alpha] f(x)| \le M_{b,\alpha}(f)(x) + 2b^-(x)M_\alpha f(x) \tag{1.1}$$

holds (see, for example, [28]). Indeed, the commutators $M_{\alpha,b}$ and $[b, M_\alpha]$ evidently differ from each other. The maximal commutator $M_{\alpha,b}$ is both positive and sublinear, while the nonlinear commutator $[b, M_\alpha]$ does not possess either property. Many authors have intensively studied the mapping properties of commutators of maximal functions, we refer the readers to see [1–4, 11–14, 22, 24, 25] and therein references.

For a given ball B and $0 \le \alpha < n$, the fractional maximal function with respect to B of a function f is defined as follows:

$$M_{\alpha,B}(f)(x) = \sup_{B \supseteq B_0 \ni x} \frac{1}{|B_0|^{1-\frac{\alpha}{n}}} \int_{B_0} |f(y)|dy,$$

where the supremum is taken over all balls B_0 with $B_0 \subseteq B$ and $B_0 \ni x$. Also, we define $M_B = M_{0,B}$ for $\alpha = 0$.

The space of functions with bounded mean oscillation, denoted as $BMO(\mathbb{R}^n)$, was introduced by John and Nirenberg [17].

Definition 1.1. The space $BMO(\mathbb{R}^n)$ consists of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_B \frac{1}{|B|}\int_B |f(x) - f_B|dx < \infty,$$

where the supremum is taken over all balls in $\mathbb{R}^n$.

Let $0 < p < \infty$, the Lebesgue space $L^p(\mathbb{R}^n)$ consists of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n)$ that satisfy the following condition:

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, {\rm d}x\right)^{\frac{1}{p}} < \infty.$$

We also need to review the decreasing rearrangement of a real function f. For $s > 0$ and $t > 0$, we define the distribution function d_f and the rearrangement function f^* as follows:

$$d_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad f^*(t) = \inf\{s > 0 : d_f(s) \le t\}.$$

We will now revisit the definition of Lorentz spaces.

Definition 1.2 ([18])**.** Given a measurable function f on $\mathbb{R}^n$ and $0 < q, r \le \infty$, we define

$$\|f\|_{L^{q,r}(\mathbb{R}^n)} := \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{q}} f^*(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}}, & \text{if } r < \infty, \\ \sup_{t>0} t^{\frac{1}{q}} f^*(t), & \text{if } r = \infty. \end{cases}$$

Thus, the Lorentz space $L^{q,r}(\mathbb{R}^n)$ consists of all functions f for which $\|f\|_{L^{q,r}(\mathbb{R}^n)} < \infty$.

Remark 1.3. If we set $r = q$, then the Lorentz space $L^{q,r}(\mathbb{R}^n)$ corresponds to the Lebesgue space $L^q(\mathbb{R}^n)$. For a ball B, we define $\|f\|_{L^{q,r}(B)} = \|f\chi_B\|_{L^{q,r}(\mathbb{R}^n)}$.

The Morrey-Lorentz spaces are defined as follows.

Definition 1.4 ([21])**.** Let $1 < q < \infty, 1 \le r \le \infty$ and $0 < \lambda \le \frac{n}{q}$. For any measurable function f, we define the Morrey-Lorentz space $L^{q,r}_\lambda(\mathbb{R}^n)$ as follows:

$$L^{q,r}_\lambda(\mathbb{R}^n) = \left\{f : \|f\|_{L^{q,r}_\lambda(\mathbb{R}^n)} = \sup_B |B|^{\frac{\lambda}{n}-\frac{1}{q}}\|f\|_{L^{q,r}(B)} < \infty\right\}.$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Remark 1.5. If we set $r = q$, then the Morrey-Lorentz $L^{q,r}_\lambda(\mathbb{R}^n)$ becomes the Morrey space $L^q_\lambda(\mathbb{R}^n)$. When $\lambda = \frac{n}{q}$, then the Morrey-Lorentz $L^{q,r}_\lambda(\mathbb{R}^n)$ corresponds to the Lorentz space $L^{q,r}(\mathbb{R}^n)$.

We can express our first result as follows.

Theorem 1.6. *Let $0 \le \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}, 0 < \mu \le \frac{n}{t}$, $0 < r, u \le \infty$, $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{r}{u} = \frac{\mu}{\lambda}$, then the subsequent statements hold equivalently:*

(T1) $b \in BMO(\mathbb{R}^n)$.

(T2) *$M_{\alpha,b}$ is bounded from $L^{q,r}_\lambda(\mathbb{R}^n)$ to $L^{t,u}_\mu(\mathbb{R}^n)$.*

(T3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - b_B)\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \le C. \tag{1.2}$$

(T4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|}\int_B |b(x) - b_B|dx \le C. \tag{1.3}$$

If we choose $r = q$, then the following corollary can be derived.

Corollary 1.7. *Let $0 \le \alpha < n$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}, 0 < \mu \le \frac{n}{t}$, $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{\mu}{\lambda}$, then the subsequent statements hold equivalently:*

(C1) $b \in BMO(\mathbb{R}^n)$.

(C2) $M_{\alpha,b}$ *is bounded from* $L^q_\lambda(\mathbb{R}^n)$ *to* $L^t_\mu(\mathbb{R}^n)$.

(C3) *There is a constant* $C > 0$ *such that*

$$\sup_B \frac{\|(b - b_B)\chi_B\|_{L^t_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^t_\mu(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx \le C.$$

If we set $\lambda = \frac{n}{q}$, then we arrive at the following conclusion.

Corollary 1.8. *Let $0 \le \alpha < n$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $1 < q, t < \infty$, $0 < r, u \le \infty$, $\frac{1}{q} - \frac{1}{t} = \frac{\alpha}{n}$ and $\frac{q}{t} = \frac{r}{u}$, then the subsequent statements hold equivalently:*

(C1) $b \in BMO(\mathbb{R}^n)$.

(C2) $M_{\alpha,b}$ *is bounded from* $L^{q,r}(\mathbb{R}^n)$ *to* $L^{t,u}(\mathbb{R}^n)$.

(C3) *There is a constant* $C > 0$ *such that*

$$\sup_B \frac{\|(b - b_B)\chi_B\|_{L^{t,u}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx \le C.$$

Here, we present our second result.

Theorem 1.9. *Let $0 \le \alpha < n$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}, 0 < \mu \le \frac{n}{t}$, $0 < r, u \le \infty$, $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{r}{u} = \frac{\mu}{\lambda}$, then the subsequent statements hold equivalently:*

(T1) $b \in BMO(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(T2) $[b, M_\alpha]$ *is bounded from* $L^{q,r}_\lambda(\mathbb{R}^n)$ *to* $L^{t,u}_\mu(\mathbb{R}^n)$.

(T3) *There is a constant* $C > 0$ *such that*

$$\sup_B \frac{\|\left(b - M_B(b)\right)\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \le C. \tag{1.4}$$

(T4) *There is a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - M_B(b)(x)| dx \le C. \tag{1.5}$$

If we take $r = q$, then we can get the following conclusion.

Corollary 1.10. *Let $0 \le \alpha < n$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}, 0 < \mu \le \frac{n}{t}$, $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{\mu}{\lambda}$, then the subsequent statements hold equivalently:*

(C1) $b \in BMO(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(C2) $[b, M_\alpha]$ *is bounded from* $L^q_\lambda(\mathbb{R}^n)$ *to* $L^t_\mu(\mathbb{R}^n)$.

(C3) *There is a constant* $C > 0$ *such that*

$$\sup_B \frac{\|\left(b - M_B(b)\right)\chi_B\|_{L^t_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^t_\mu(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - M_B(b)(x)| dx \le C.$$

If we take $\lambda = \frac{n}{q}$, then the following result holds.

Corollary 1.11. *Let $0 \leq \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < q, t < \infty$, $0 < r, u \leq \infty$, $\frac{1}{q} - \frac{1}{t} = \frac{\alpha}{n}$ and $\frac{q}{t} = \frac{r}{u}$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(C2) *$[b, M_\alpha]$ is bounded from $L^{q,r}(\mathbb{R}^n)$ to $L^{t,u}(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\| (b - M_B(b))\, \chi_B \|_{L^{t,u}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}(\mathbb{R}^n)}} \leq C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - M_B(b)(x)| dx \leq C.$$

Next, our third result is as follows.

Theorem 1.12. *Let $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $0 < u \leq \infty$, $1 < t < \infty$ and $0 < \mu \leq \frac{n}{t}$, then the subsequent statements hold equivalently:*

(T1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(T2) *$[b, M^\sharp]$ is bounded on $L^{t,u}_\mu(\mathbb{R}^n)$.*

(T3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - 2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \leq C. \tag{1.6}$$

(T4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - 2M^\sharp(b\chi_B)(x)| dx \leq C. \tag{1.7}$$

If we take $r = q$, then the following conclusion holds.

Corollary 1.13. *Let $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < t < \infty, 0 < \mu \leq \frac{n}{t}$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(C2) *$[b, M^\sharp]$ is bounded on $L^t_\mu(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - 2M^\sharp(b\chi_B))\chi_B\|_{L^t_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^t_\mu(\mathbb{R}^n)}} \leq C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - 2M^\sharp(b\chi_B)(x)| dx \leq C.$$

If we take $\lambda = \frac{n}{q}$, then the following result can be obtained.

Corollary 1.14. *Let $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < t < \infty$, $0 < u \leq \infty$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(C2) *$[b, M^\sharp]$ is bounded on $L^{t,u}(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - 2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}(\mathbb{R}^n)}} \leq C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - 2M^\sharp(b\chi_B)(x)| dx \leq C.$$

2. Preliminaries

To demonstrate our main results, we will present several important notions and known results in the section.

First, we must introduce the predual spaces of Morrey-Lorentz spaces.

Definition 2.1 ([7]). Let $1 < q < \infty, 1 \le r \le \infty$ and $\beta > 0$. A function $b(x)$ is called a (q, r, β)-block, if there exists a ball B in $\mathbb{R}^n$, such that

$$\operatorname{supp}(b) \subset B(x_0, r), \quad \|b\|_{L^{q,r}(B)} \le |B|^{\frac{1}{q}-\frac{\beta}{n}}$$

Next, we define the space $\mathscr{B}_{\beta}^{q,r}(\mathbb{R}^n)$ using (q, r, β)-blocks.

Definition 2.2 ([7]). Let $1 < q < \infty, 1 \le r \le \infty$ and $\frac{n}{q} \le \beta < n$. The space $\mathscr{B}_{\beta}^{q,r}(\mathbb{R}^n)$ is defined as follows:

$$\mathscr{B}_{\beta}^{q,r}(\mathbb{R}^n) = \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : g = \sum_{j=1}^{\infty} m_j b_j, \{b_j\}_{j\ge 1} \text{ are } (q,r,\beta)\text{-blocks and } \sum_{j=1}^{\infty} |m_j| < \infty \right\}.$$

Lemma 2.3 ([7]). *Let $1 < q < \infty, 1 \le r \le \infty$, and $0 < \lambda \le \frac{n}{q}$. Then*

$$L_{\lambda}^{q,r}(\mathbb{R}^n) = \left(\mathscr{B}_{n-\lambda}^{q',r'}(\mathbb{R}^n)\right)' \text{ and } L_{\lambda}^{q,r}(\mathbb{R}^n)' = \mathscr{B}_{n-\lambda}^{q',r'}(\mathbb{R}^n)$$

Lemma 2.4 ([7]). *Let $1 < q < \infty$, $1 \le r \le \infty$, $0 < \lambda \le \frac{n}{q}$ and $\frac{n}{q} \le \beta < n$. Then*

$$\|\chi_B\|_{L_{\lambda}^{q,r}(\mathbb{R}^n)} \approx |B|^{\frac{\lambda}{n}} \text{ and } \|\chi_B\|_{\mathscr{B}_{\beta}^{q,r}(\mathbb{R}^n)} \approx |B|^{\frac{\beta}{n}}.$$

Lemma 2.5 ([7]). *Let $1 < q, q', r, r' < \infty$ and $0 < \lambda \le \frac{n}{q}$. Assume that $f \in L_{\lambda}^{q,r}(\mathbb{R}^n)$ and $g \in \mathscr{B}_{n-\lambda}^{q',r'}(\mathbb{R}^n)$. Then the following statement is true:*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \lesssim \|f\|_{L_{\lambda}^{q,r}(\mathbb{R}^n)} \|g\|_{\mathscr{B}_{n-\lambda}^{q',r'}(\mathbb{R}^n)}$$

Similarly to [15, Proposition 3], we obtain the following conclusion, the proof of which requires only slight modifications; thus, we omit the details.

Lemma 2.6. *Let $0 \le \alpha < n$, $0 < r, u \le \infty$, $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}$ and $0 < \mu \le \frac{n}{t}$. Suppose that $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{r}{u} = \frac{\mu}{\lambda}$. Then for $f \in L_{\lambda}^{q,r}(\mathbb{R}^n)$,*

$$\|M_{\alpha} f\|_{L_{\mu}^{t,u}(\mathbb{R}^n)} \lesssim \|f\|_{L_{\lambda}^{q,r}(\mathbb{R}^n)}.$$

Lemma 2.7 ([10]). *Let $0 \le \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Then, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, there is a constant C such that*

$$M_{b,\alpha} f(x) \le C\|b\|_{BMO(\mathbb{R}^n)} (M(M_{\alpha} f)(x) + M_{\alpha}(Mf)(x)).$$

Lemma 2.8 ([26]). *Let $0 \le \alpha < n$, B be a ball in $\mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then, for any $x \in B$, it holds that:*

$$M_{\alpha}(f\chi_B)(x) = M_{\alpha,B}(f)(x).$$

3. Proofs of main results

Proof of Theorem 1.6. (T1) $\Rightarrow$ (T2): Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Lemma 2.6 with Lemma 2.7 deduces that

$$\begin{aligned} \|M_{\alpha,b}(f)\|_{L_{\mu}^{t,u}(\mathbb{R}^n)} &\le C\|b\|_{BMO(\mathbb{R}^n)} \|(M(M_{\alpha} f)(x) + M_{\alpha}(Mf)(x))\|_{L_{\mu}^{t,u}(\mathbb{R}^n)} \\ &\le C\|b\|_{BMO(\mathbb{R}^n)} (\|M_{\alpha} f\|_{L_{\mu}^{t,u}(\mathbb{R}^n)} + \|Mf\|_{L_{\lambda}^{q,r}(\mathbb{R}^n)}) \\ &\le C\|b\|_{BMO(\mathbb{R}^n)} \|f\|_{L_{\lambda}^{q,r}(\mathbb{R}^n)}. \end{aligned}$$

Thus, we conclude that $M_{\alpha,b}$ is bounded from $L^{q,r}_{\lambda}(\mathbb{R}^n)$ to $L^{t,u}_{\mu}(\mathbb{R}^n)$.

(T2) $\Rightarrow$ (T3): For a given ball $B \subset \mathbb{R}^n$ and $x \in B$, we obtain

$$\begin{aligned}|b(x)-b_B| &\leq \frac{1}{|B|}\int_B |b(x)-b(y)|dy \\ &= \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B |b(x)-b(y)|\chi_B(y)dy \\ &\leq |B|^{-\frac{\alpha}{n}} M_{\alpha,b}(\chi_B)(x).\end{aligned}$$

Since $M_{\alpha,b}$ is bounded from $L^{q,r}_{\lambda}(\mathbb{R}^n)$ to $L^{t,u}_{\mu}(\mathbb{R}^n)$, then using Lemma 2.4 and the condition $\lambda-\alpha=\mu$, we conclude that

$$\begin{aligned}\frac{\|(b-b_B)\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n))}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} &\leq \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\|M_{\alpha,b}(\chi_B)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\leq C\frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\|\chi_B\|_{L^{q,r}_{\lambda}(\mathbb{R}^n))}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\leq C,\end{aligned}$$

which deduces that (1.2) holds since the ball $B \subset \mathbb{R}^n$ is arbitrary.

(T3) $\Rightarrow$ (T4): Assume that (1.2) is true, we will show (1.3). For a given ball B, by applying Lemma 2.4 and Lemma 2.5, we can derive

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-b_B|dx &\leq C\frac{1}{|B|}\|(b-b_B)\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)} \\ &\leq C\frac{\|(b-b_B)\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\leq C.\end{aligned}$$

(T4) $\Rightarrow$ (T1): It follows from Definition 1.1 directly, thus we omit the details.

This finishes the proof of Theorem 1.6. □

Proof of Theorem 1.9. (T1) $\Rightarrow$ (T2): Suppose that $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^{\infty}(\mathbb{R}^n)$. By (1.1), Lemma 2.6 and Lemma 2.7, we have

$$\begin{aligned}\|[b,M_{\alpha}](f)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)} &\leq \|M_{b,\alpha}(f)+2b^-M_{\alpha}(f)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)} \\ &\leq \|M_{b,\alpha}(f)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}+\|2b^-M_{\alpha}(f)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{q,r}_{\lambda}(\mathbb{R}^n)}+\|b^-\|_{L^{\infty}(\mathbb{R}^n)}\|f\|_{L^{q,r}_{\lambda}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{L^{q,r}_{\lambda}(\mathbb{R}^n)}.\end{aligned}$$

Thus, we show that $[b,M_{\alpha}]$ is bounded from $L^{q,r}_{\lambda}(\mathbb{R}^n)$ to $L^{t,u}_{\mu}(\mathbb{R}^n)$.

(T2) $\Rightarrow$ (T3): We will divide the proof into two cases depending on the value of α.

Case 1. Let $0<\alpha<n$. For a given ball B,

$$\begin{aligned}\frac{\|(b-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} &\leq \frac{\|(b-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)\,)\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\quad+\frac{\|(|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &:= I+II.\end{aligned}$$

For I. For any $x \in B$, the definition of $M_{\alpha,B}$ implies that

$$M_{\alpha,B}(\chi_B)(x)=|B|^{\frac{\alpha}{n}}. \tag{3.1}$$

For any $x \in B$, Lemma 2.8 indicates that,

$$M_\alpha(\chi_B)(x) = M_{\alpha,B}(\chi_B)(x) = |B|^{\frac{\alpha}{n}} \text{ and } M_\alpha(b\chi_B)(x) = M_{\alpha,B}(b)(x).$$

Therefore, we have

$$\begin{aligned} b(x) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(x) &= |B|^{-\frac{\alpha}{n}} (b(x)|B|^{\frac{\alpha}{n}} - M_{\alpha,B}(b)(x)) \\ &= |B|^{-\frac{\alpha}{n}} (b(x) M_\alpha(\chi_B)(x) - M_\alpha(b\chi_B)(x)) \\ &= |B|^{-\frac{\alpha}{n}} [b, M_\alpha](\chi_B)(x). \end{aligned}$$

Since $[b, M_\alpha]$ is bounded from $L^{q,r}_\lambda(\mathbb{R}^n)$ to $L^{t,u}_\mu(\mathbb{R}^n)$, then combining Lemma 2.4 with the condition $\lambda - \alpha = \mu$ deduces that

$$\begin{aligned} I &= \frac{\|(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &= \frac{1}{|B|^{\frac{\alpha}{n}}} \frac{\|[b, M_\alpha](\chi_B)\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &\le C \frac{1}{|B|^{\frac{\alpha}{n}}} \frac{\|\chi_B\|_{L^{q,r}_\lambda(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &\le C. \end{aligned}$$

For II. Similar to (3.1), by using Lemma 2.8 and for any $x \in B$,

$$M_B(\chi_B)(x) = \chi_B(x),$$

we deduce that

$$M(\chi_B)(x) = \chi_B(x) \text{ and } M(b\chi_B)(x) = M_B(b)(x). \tag{3.2}$$

Thus, Combining (3.1) with (3.2) implies that

$$\begin{aligned} \left||B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(x) - M_B(b)(x)\right| &\le |B|^{-\frac{\alpha}{n}} |M_\alpha(b\chi_B)(x) - |b(x)| M_\alpha(\chi_B)(x)| \\ &\quad + |B|^{-\frac{\alpha}{n}} ||b(x)| M_\alpha(\chi_B)(x) - M_\alpha(\chi_B)(x) M(b\chi_B)(x)| \\ &= |B|^{-\frac{\alpha}{n}} |M_\alpha(|b|\chi_B)(x) - |b(x)| M_\alpha(\chi_B)(x)| \\ &\quad + |B|^{-\frac{\alpha}{n}} M_\alpha(\chi_B)(x) ||b(x)| M(\chi_B)(x) - M(b\chi_B)(x)| \\ &= |B|^{-\frac{\alpha}{n}} |[|b|, M_\alpha](\chi_B)(x)| + |[|b|, M](\chi_B)(x)|. \end{aligned}$$

Since $[b, M_\alpha]$ is bounded from $L^{q,r}_\lambda(\mathbb{R}^n)$ to $L^{t,u}_\mu(\mathbb{R}^n)$. Then, by applying Lemma 2.4, we get

$$\begin{aligned} II &\le \frac{\|(|B|^{-\frac{\alpha}{n}} |[|b|, M_\alpha](\chi_B)| + |[|b|, M](\chi_B)|)\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &\lesssim \frac{1}{|B|^{\frac{\alpha}{n}}} \frac{\|\chi_B\|_{L^{q,r}_\lambda(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} + \frac{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &\le C. \end{aligned}$$

This deduces that the desired estimate

$$\frac{\|(b - M_B(b))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \le C,$$

which concludes that (1.4) holds.

Case 2. Let $\alpha = 0$. For a given ball B and $x \in B$, using (3.2), we obtain

$$b(x) - M_B(b)(x) = b(x) M(\chi_B)(x) - M(b\chi_B)(x) = [b, M](\chi_B)(x).$$

Suppose that $[b, M]$ is bounded from $L^{q,r}_{\lambda}(\mathbb{R}^n)$ to $L^{t,u}_{\mu}(\mathbb{R}^n)$, then by applying Lemma 2.4, we have

$$\begin{aligned}\frac{\|(b-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} &= \frac{\|[b,M](\chi_B)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}\\ &\leq C\frac{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}\\ &\leq C,\end{aligned}$$

which implies that (1.4).

(T3) $\Rightarrow$ (T4): Assume that (1.4) holds, then for a given ball B, by Lemma 2.5, we have

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-M_B(b)(x)|\,dx &\leq C\frac{1}{|B|}\,\|(b-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}\,\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)}\\ &\leq C\frac{1}{|B|}\,\frac{\|(b-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}\\ &\leq C,\end{aligned}$$

where the constant C does not depend on B. This deduces that (1.5).

(T4) $\Rightarrow$ (T1): To prove $b \in BMO(\mathbb{R}^n)$, we only need to demonstrate that there exists a constant $C > 0$ such that, for a given ball B,

$$\frac{1}{|B|}\int_B |b(x)-b_B|dx \leq C.$$

For a given ball B, let $E = \{x \in B : b(x) \leq b_B\}$ and $F = \{x \in B : b(x) > b_B\}$, then we get

$$\int_E |b(x)-b_B|dx = \int_F |b(x)-b_B|dx. \tag{3.3}$$

As $b(x) \leq b_B \leq M_B(b)(x)$ for any $x \in E$, we obtain

$$|b(x)-b_B| \leq |b(x)-M_B(b)(x)|. \tag{3.4}$$

Combining (3.3) with (3.4) deduces that

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-b_B|dx &= \frac{2}{|B|}\int_E |b(x)-b_B|dx\\ &\leq \frac{2}{|B|}\int_E |b(x)-M_B(b)(x)|dx\\ &\leq \frac{2}{|B|}\int_B |b(x)-M_B(b)(x)|dx\\ &\leq C.\end{aligned}$$

Thus, we deduce that $b \in BMO(\mathbb{R}^n)$.

Next, we aim to prove that $b^- \in L^{\infty}(\mathbb{R}^n)$. Note that for any $y \in B$, we have $0 \leq b^+(y) \leq |b(y)| \leq M_B(b)(y)$, then

$$0 \leq b^-(y) \leq M_B(b)(y) - b^+(y) + b^-(y) = M_B(b)(y) - b(y).$$

Furthermore, for a given ball B, we get

$$\begin{aligned}\frac{1}{|B|}\int_B b^-(y)dy &\leq \frac{1}{|B|}\int_B \left(M_B(b)(y)-b(y)\right)dy\\ &= \frac{1}{|B|}\int_B |b(y)-M_B(b)(y)|\,dy\\ &\leq C.\end{aligned}$$

Let $|B| \to 0$ with $x \in B$. By applying Lebesgue's differentiation theorem, we deduce that

$$0 \le b^-(x) = \lim_{|B|\to 0} \frac{1}{|B|} \int_B b^-(y)dy \le C.$$

Hence, we establish that $b^- \in L^\infty(\mathbb{R}^n)$.

We have now completed the proof of Theorem 1.9. □

Proof of Theorem 1.12. (T1) ⇒ (T2): Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$, for a given ball $B \subset \mathbb{R}^n$, the estimate below was established in [27]:

$$|[|b|, M^\sharp]f(x)| \le 2M_{|b|}f(x).$$

Noting that $|b| - b = 2b^-$, it follows from the definition of $[b, M^\sharp]$ that,

$$\begin{aligned}
&|[b, M^\sharp]f(x) - [|b|, M^\sharp]f(x)| \\
&\le |M^\sharp(bf)(x) - M^\sharp(|b|f)(x)| + ||b(x)|M^\sharp(f)(x) - b(x)M^\sharp f(x)| \\
&\le |M^\sharp((b - |b|)f)(x)| + 2b^-(x)M^\sharp f(x) \\
&\le M^\sharp(2b^- f)(x) + 2b^-(x)M^\sharp f(x).
\end{aligned}$$

Combined with previous estimates and $M^\sharp(f) \le 2M(f)$, for any $x \in \mathbb{R}^n$, we obtain

$$\begin{aligned}
|[b, M^\sharp](f)(x)| &\le |[b, M^\sharp]f(x) - [|b|, M^\sharp]f(x)| + |[|b|, M^\sharp]f(x)| \\
&\le M^\sharp(2b^- f)(x) + 2b^-(x)M^\sharp(f)(x) + |[|b|, M^\sharp]f(x)|, \\
&\le 2M(2b^- f)(x) + 4b^-(x)M(f)(x) + 2M_{|b|}f(x).
\end{aligned}$$

Since $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $|b| \in \mathrm{BMO}(\mathbb{R}^n)$. Based on Lemma 2.6 and Theorem 1.6, we find that

$$\|[b, M^\sharp](f)\|_{L^{t,u}_\mu(\mathbb{R}^n)} \le C\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{t,u}_\mu(\mathbb{R}^n)},$$

which implies that $[b, M^\sharp]$ is bounded on $L^{t,u}_\mu(\mathbb{R}^n)$.

(T2) ⇒ (T3): Take B as a fixed ball and B_1 as a different ball. By the inequality $4ac \le (a + c)^2$, we can see that

$$\begin{aligned}
&\frac{1}{|B_1|}\int_{B_1} |\chi_B(x) - (\chi_B)_{B_1}|dx \\
&= \frac{1}{|B_1|}\Big\{\int_{B_1\setminus B} |\chi_B(x) - (\chi_B)_{B_1}|dx + \int_{B_1\cap B} |\chi_B(x) - (\chi_B)_{B_1}|dx\Big\} \\
&= \frac{1}{|B_1|}\Big\{\int_{B_1\setminus B} |(\chi_B)_{B_1}|dx + \int_{B_1\cap B} |1 - (\chi_B)_{B_1}|dx\Big\} \\
&= \frac{1}{|B_1|}\Big\{\int_{B_1\setminus B} \Big|\frac{1}{|B_1|}\int_{B_1\cap B} \chi_B(y)dy\Big|dx \\
&+ \int_{B_1\cap B} \Big|\frac{1}{|B_1|}\int_{B_1} \chi_{B_1}(y)dy - \frac{1}{|B_1|}\int_{B_1} \chi_B(y)\cdot\chi_{B_1}(y)dy\Big|dx\Big\} \\
&= \frac{1}{|B_1|}\Big\{\frac{|B_1\cap B||B_1\setminus B|}{|B_1|} + \frac{1}{|B_1|}\int_{B_1\cap B}\Big|\int_{B_1} \chi_{B_1}(y)(1 - \chi_B(y))dy\Big|dx\Big\} \\
&= \frac{1}{|B_1|^2}\Big\{|B_1\cap B||B_1\setminus B| + |B_1\cap B||B_1\setminus B|\Big\} \\
&= \frac{2|B_1\cap B||B_1\setminus B|}{(|B_1\cap B| + |B_1\setminus B|)^2} \le \frac{1}{2}.
\end{aligned} \tag{3.5}$$

Moreover, for $x \in B$, we can find a ball B_0 that contains B and satisfies $|B_0| = 2|B|$. Then, using (3.5) and $|B_0 \setminus B| = |B_0 \cap B| = |B|$, we conclude that

$$\frac{1}{|B_0|}\int_{B_0} |\chi_B(x) - (\chi_B)_{B_0}|dx = \frac{2|B_0 \cap B||B_0 \setminus B|}{(|B_0 \cap B| + |B_0 \setminus B|)^2} = \frac{1}{2}.$$

Furthermore, we have

$$(M^\sharp(\chi_B)\chi_B)(x) = \sup_{B_1 \ni x} \frac{1}{|B_1|}\int_{B_1} |\chi_B(y) - (\chi_B)_{B_1}|dy = \frac{1}{2} = \frac{1}{2}\chi_B(x).$$

Then, we can get

$$\begin{aligned}
\|(b - 2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)} &= \Big\|2\Big(\frac{1}{2}b\chi_B - M^\sharp(b\chi_B)\Big)\chi_B\Big\|_{L^{t,u}_\mu(\mathbb{R}^n)} \\
&= \|2(bM^\sharp(\chi_B)\chi_B - M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)} \\
&= \|2(bM^\sharp(\chi_B) - M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)} \\
&\leq \|2[b, M^\sharp](\chi_B)\|_{L^{t,u}_\mu(\mathbb{R}^n)} \\
&\leq C\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)},
\end{aligned}$$

where the constant C does not depend on B. This deduces that (1.6).

(T3) $\Rightarrow$ (T4): For a ball $B \subset \mathbb{R}^n$ and $x \in B$, we will show that $|b_B| \leq 2M^\sharp(b\chi_B)(x)$. Take $x \in B$ and select a ball B_1 that includes B with the property that $|B_1| = 2|B|$. Thus,

$$\begin{aligned}
\frac{1}{2|B|}\int_B |b(y) - \frac{1}{2}b_B|dy + \frac{1}{4}|b_B| &= \frac{1}{2|B|}\left(\int_B |b(y) - \frac{1}{2}b_B|dy + \frac{1}{2}|B_1 \setminus B||b_B|\right) \\
&= \frac{1}{|B_1|}\int_{B_1} |b\chi_B(y) - (b\chi_B)_{B_1}|dy \\
&\leq M^\sharp(b\chi_B)(x).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|b_B| &\leq \frac{1}{|B|}\int_B |b(y) - \frac{1}{2}b_B|dy + \frac{1}{|B|}\int_B |\frac{1}{2}b_B|dy \\
&= \frac{1}{|B|}\int_B |b(y) - \frac{1}{2}b_B|dy + \frac{1}{2}|b_B|.
\end{aligned}$$

Thus, for $x \in B$, we obtain

$$|b_B| \leq 2M^\sharp(b\chi_B)(x). \tag{3.6}$$

Next, we will show that $b \in BMO(\mathbb{R}^n)$. To do this, let $E = \{x \in B : b(x) \leq b_B\}$ and $F = \{x \in B : b(x) > b_B\}$, we then obtain

$$\int_E |b(x) - b_B|dx = \int_F |b(x) - b_B|dx.$$

Since $b(x) \leq b_B \leq |b_B| \leq 2M^\sharp(b\chi_B)(x)$ for any $x \in E$, then

$$|b(x) - b_B| \leq |b(x) - 2M^\sharp(b\chi_B)(x)|.$$

Using Lemma 2.4 and Lemma 2.5, we get

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-b_B|dx &= \frac{1}{|B|}\int_{E\cup F} |b(x)-b_B|dx\\ &= \frac{2}{|B|}\int_E |b(x)-b_B|dx\\ &\le \frac{2}{|B|}\int_E |b(x)-2M^\sharp(b\chi_B)(x)|dx\\ &\le \frac{2}{|B|}\int_B |b(x)-2M^\sharp(b\chi_B)(x)|dx\\ &\le \frac{C}{|B|}\|(b-2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)}\\ &\le \frac{C}{|B|}\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)}\\ &\le C,\end{aligned}$$

which deduces that $b\in BMO(\mathbb{R}^n)$. We shall now prove that b^- is in $L^\infty(\mathbb{R}^n)$. By (3.6), for $x\in B$, we have

$$|b_B|-b^+(x)+b^-(x)=|b_B|-b(x)\le 2M^\sharp(b\chi_B)(x)-b(x).$$

Therefore,

$$\begin{aligned}|b_B|-\frac{1}{|B|}\int_B b^+(x)dx+\frac{1}{|B|}\int_B b^-(x)dx &= \frac{1}{|B|}\int_B(|b_B|-b^+(x)+b^-(x))dx\\ &\le \frac{1}{|B|}\int_B(2M^\sharp(b\chi_B)(x)-b(x))dx\\ &\le \frac{1}{|B|}\int_B |b(x)-2M^\sharp(b\chi_B)(x)|dx.\end{aligned}\tag{3.7}$$

Besides, combining Lemma 2.4 with Lemma 2.5, we can get

$$\begin{aligned}&\frac{1}{|B|}\int_B |b(x)-2M^\sharp(b\chi_B)(x)|dx\\ &\le \frac{C}{|B|}\|(b-2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)}\\ &\le \frac{C}{|B|}\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)}\\ &\le C.\end{aligned}$$

Combining this inequality with (3.7), we deduce that

$$|b_B|-\frac{1}{|B|}\int_B b^+(x)dx+\frac{1}{|B|}\int_B b^-(x)dx\le C.$$

Let $|B|$ tend to 0 with $x\in B$, it follows from Lebesgue's differentiation theorem that,

$$2|b^-(x)|=2b^-(x)=|b(x)|-b^+(x)+b^-(x)\le C.$$

This implies that $b^-\in L^\infty(\mathbb{R}^n)$.

Therefore, we complete the proof of Theorem 1.12. □

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